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SCHAUDER ESTIMATES FOR EQUATIONS WITH FRACTIONAL DERIVATIVES

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ABSTRACT. The equation

$$(*) \qquad D_t^\alpha(u-h_1) + D_x^\beta(u-h_2) = f, \quad 0<\alpha,\beta<1, \quad t,x\geq 0,$$

where D_t^α and D_x^β are fractional derivatives of order α and β is studied. It is shown that if $f = f(\underline{t},\underline{x})$, $h_1 = h_1(\underline{x})$, and $h_2 = h_2(\underline{t})$ are Hölder-continuous and $f(0,0)=0$, then there is a solution such that $D_t^\alpha u$ and $D_x^\beta u$ are Hölder-continuous as well. This is proved by first considering an abstract fractional evolution equation and then applying the results obtained to $(*)$. Finally the solution of $(*)$ with $f=1$ is studied.

1. Introduction

The purpose of this paper is to study the partial differential equation

$$D_t^\alpha(u-h_1) + D_x^\beta(u-h_2) = f, \tag{1}$$

on $\mathbb{R}^+ \times \mathbb{R}^+$ ($\mathbb{R}^+ \stackrel{\text{def}}{=} [0,\infty)$). The function u is the unknown and h_1, h_2, and f are given.

In (1) D_t^α and D_x^β denote the fractional derivatives of order α and β ($\in (0,1)$), see [17, p. 133], i.e.,

$$(D_t^\alpha v)(t) \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t g_{1-\alpha}(t-s)v(s)\,\mathrm{d}s, \quad t>0,$$

$$(D_t^\alpha v)(0) \stackrel{\text{def}}{=} \lim_{h\downarrow 0} \tfrac{1}{h}\int_0^h g_{1-\alpha}(h-s)v(s)\,\mathrm{d}s,$$

where

$$g_\beta(t) \stackrel{\text{def}}{=} \frac{1}{\Gamma(\beta)}t^{\beta-1}, \quad t>0, \quad \beta>0,$$

and where v is (at least) continuous and satisfies $v(0)=0$. If $w(\underline{t},\underline{x})$ is a function of two variables, then $(D_t^\alpha w)(\underline{t},\underline{x})$ is the function $(t,x)\mapsto (D_t^\alpha w(\bullet,x))(t)$ and $(D_x^\beta w)(\underline{t},\underline{x})$ is the function $(t,x)\mapsto (D_x^\beta w(t,\bullet))(x)$ (i.e., the roles of t and x are interchanged).

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From (1) and from the definition of the fractional derivative it follows that if h_1 depends on x only, and h_2 on t only (this is the case we consider), then h_1 and h_2 are boundary conditions imposed on the solution; that is, we must have

$$u(0,x) = h_1(x), \quad x \geq 0,$$
$$u(t,0) = h_2(t), \quad t \geq 0.$$

The main goal of our study is to obtain maximal regularity results in the setting of Hölder continuous functions and to obtain related Schauder estimates. The Hölder spaces arise naturally as interpolation spaces between the domains of the fractional derivative operators and the space of continuous functions.

Our study of the regularity of (1) is to some extent motivated by questions associated with the nonlinear fractional conservation law

$$D_t^\alpha(u - u_0) + D_x\sigma(u) = f, \tag{2}$$

where $\alpha \in (0,1)$ and σ is sufficiently smooth. Equations of this type have recently been employed to approximate nonlinear conservation laws, and the behavior of entropy solutions of (2) when $\alpha \uparrow 1$ has been investigated; see [4], [8], and [9]. However, the regularity of solutions of (2) remains largely unsettled. Observe that formally a solution of (2) satisfies the equation

$$D_t^\alpha(u(t,x) - u_0(x)) + a(t,x)u_x(t,x) = f(t,x), \quad t,x > 0, \tag{3}$$

where $a(t,x) = \sigma'(u(t,x))$. Equation (3) is a version of problem (1) with $\beta = 1$ and non-constant coefficients. This forces us to consider the case where $\beta = 1$ too. It will turn out that our approach will work provided the fundamental condition $\alpha + \beta < 2$ holds. However, the case where one of the exponents is at least 1 has to be dealt with in a somewhat different way. Therefore, for the sake of clarity, we restrict ourselves in this paper to the case where both α and β are strictly less than 1 and consider the other case in a subsequent paper.

A second motivation for the present work has been the results of Sinestrari, [15], and of Da Prato and Sinestrari, [7], on maximal regularity of the evolution equation

$$\tfrac{\mathrm{d}u}{\mathrm{d}t} = Au + f, \quad u(0) = u_0,$$

where $A : \mathcal{D}(A) \subset X \to X$ is closed and linear but not necessarily densely defined. Although our proofs are different and, obviously, concern (1), the formulation of our key Lemma 12 is inspired by [7, Thm. 13.5].

Our analysis combines two different approaches. The first is the method of sums introduced in [5]. Applying this method we handle data vanishing for $x = 0$ and for $t = 0$, i.e., $h_1 = h_2 = 0$. By the second method, that is, the resolvent approach for abstract equations, see [3] and [6], we handle data that do not vanish on the boundary, but where the forcing function f is a function of only one variable. In both cases we have to assume, however, that $f(0,0) = 0$. Finally we develop estimates on the fundamental solution of (1) and use these to obtain some results in the case where $f(0,0) \neq 0$.

The paper is organized as follows. In Section 2, we formulate our results on (1). In the next section we state regularity results on the more general fractional evolution equation

$$D_t^\alpha(u - u_0) + Bu = f, \tag{4}$$

in a Banach space setting. These results are proved in Section 4. In Section 5, we derive the results on (1) given in Section 2 from the abstract results in Section 3. Finally we prove the properties of the fundamental solution of (1).

2. Schauder estimates and the fundamental solution

In this section we formulate our results for equation (1) and we start by recalling some standard definitions.

Let X be a complex Banach space. We shall denote by $\mathcal{C}(\mathcal{I};X)$ the space of continuous functions: $\mathcal{I}\to X$ and by $\mathcal{B}(\mathcal{I};X)$ the space of bounded functions (here $\mathcal{I}$ is either an interval or a rectangle). We use the sup-norm in both cases. The Hölder spaces $\mathcal{C}^\gamma$, $0<\gamma<1$, are defined by

$$\mathcal{C}^\gamma([0,T];X) \overset{\text{def}}{=} \left\{ f\in\mathcal{C}([0,T];X) \;\middle|\; \sup_{0\le s<t\le T}\frac{\|f(t)-f(s)\|_X}{|t-s|^\gamma}<\infty \right\}$$

with

$$\|f\|_{\mathcal{C}^\gamma} \overset{\text{def}}{=} \sup_{t\in[0,T]}\|f(t)\|_X + \sup_{0\le s<t\le T}\frac{\|f(t)-f(s)\|_X}{|t-s|^\gamma}.$$

The little Hölder spaces h^γ are defined by

$$h^\gamma([0,T];X) \overset{\text{def}}{=} \left\{ f\in\mathcal{C}^\gamma([0,T];X) \;\middle|\; \limsup_{\substack{\delta\downarrow 0\\ t,s\in[0,T],0<|t-s|\le\delta}}\frac{\|f(t)-f(s)\|_X}{|t-s|^\gamma}=0 \right\}.$$

The Hölder spaces of scalar-valued functions in two variables can be defined by

$$\begin{aligned}\mathcal{C}^{\mu,\nu}(Q;\mathbb{C}) \overset{\text{def}}{=} \big\{ f\in\mathcal{C}(Q;\mathbb{C}) \;|\; & \\ & |f(x_1,y_1)-f(x_2,y_2)|\le M\big(|x_1-x_2|^\mu+|y_1-y_2|^\nu\big),\\ & (x_1,y_1),(x_2,y_2)\in Q, \text{for some constant } M \big\},\end{aligned}$$

where Q is a compact subset of $\mathbb{R}^2$ and $\mu,\nu\in(0,1)$. The norm in $\mathcal{C}^{\mu,\nu}(Q;\mathbb{C})$ is given by

$$\|f\|_{\mathcal{C}^{\mu,\nu}(Q)} \overset{\text{def}}{=} \sup_{\substack{(x_1,y_1),(x_2,y_2)\in Q\\(x_1,y_1)\ne(x_2,y_2)}}\frac{|f(x_1,y_1)-f(x_2,y_2)|}{|x_1-x_2|^\mu+|y_1-y_2|^\nu} + \sup_{(x,y)\in Q}|f(x,y)|.$$

We are now in a position to state our main results about equation (1).

Theorem 1. *Assume that α, $\beta\in(0,1)$, $\mu\in(0,\alpha)$, $\nu\in(0,\beta)$ and that for some τ, $\xi>0$, $\alpha'\in(\alpha,1)$, and $\beta'\in(\beta,1)$ we have*

(i) $f\in\mathcal{C}^{\mu,\nu}([0,\tau]\times[0,\xi];\mathbb{C})$ *with* $f(0,0)=0$;
(ii) $h_1\in\mathcal{C}^{\beta'}([0,\xi];\mathbb{C})$ *with* $h_1(0)=(D_x^\beta h_1)(0)=0$;
(iii) $h_2\in\mathcal{C}^{\alpha'}([0,\tau];\mathbb{C})$ *with* $h_2(0)=(D_t^\alpha h_2)(0)=0$;
(iv) $f(0,\underline{x})-(D_x^\beta h_1)(\underline{x})\in\mathcal{C}^{\max\{\nu,\frac{\mu\beta}{\alpha}\}}([0,\xi];\mathbb{C})$;
(v) $f(\underline{t},0)-(D_t^\alpha h_2)(\underline{t})\in\mathcal{C}^{\max\{\mu,\frac{\nu\alpha}{\beta}\}}([0,\tau];\mathbb{C})$.

Let $h_1^\square(t,x)\overset{\text{def}}{=}h_1(x)$ and $h_2^\square(t,x)\overset{\text{def}}{=}h_2(t)$ for $(t,x)\in[0,\tau]\times[0,\xi]$. Then there is a unique function $u\in\mathcal{C}([0,\tau]\times[0,\xi];\mathbb{C})$ satisfying

(a) $u(0,x)=h_1(x)$, $x\in[0,\xi]$;
(b) $u(t,0)=h_2(t)$, $t\in[0,\tau]$;
(c) $D_t^\alpha(u-h_1^\square)\in\mathcal{C}^{\mu,\nu}([0,\tau]\times[0,\xi];\mathbb{C})$;
(d) $D_x^\beta(u-h_2^\square)\in\mathcal{C}^{\mu,\nu}([0,\tau]\times[0,\xi];\mathbb{C})$;

(e) *For all* $(t,x) \in [0,\tau] \times [0,\xi]$,

$$\big(D_t^\alpha (u - h_1^\square)\big)(t,x) + \big(D_x^\beta (u - h_2^\square)\big)(t,x) = f(t,x).$$

Moreover, there is a constant M *such that*

$$(5)\quad \begin{aligned} &\left\| D_t^\alpha (u - h_1^\square)\right\|_{\mathcal{C}^{\mu,\nu}([0,\tau]\times[0,\xi])} + \left\| D_x^\beta (u - h_2^\square)\right\|_{\mathcal{C}^{\mu,\nu}([0,\tau]\times[0,\xi])} \\ &\quad \le M\Bigg(\|f\|_{\mathcal{C}^{\mu,\nu}([0,\tau]\times[0,\xi])} + \left\| f(\underline{t},0) - (D_t^\alpha h_2)(\underline{t})\right\|_{\mathcal{C}^{\max\{\mu,\frac{\nu\alpha}{\beta}\}}([0,\tau])} \\ &\qquad + \left\| f(0,\underline{x}) - (D_x^\beta h_1)(\underline{x})\right\|_{\mathcal{C}^{\max\{\nu,\frac{\mu\beta}{\alpha}\}}([0,\xi])}\Bigg). \end{aligned}$$

The constant M *can be written as* $M = M_0 \dfrac{\max\{1,\tau^\mu,\tau^{\frac{\nu\alpha}{\beta}},\xi^\nu,\xi^{\frac{\mu\beta}{\alpha}}\}}{\min\{1,\tau^\mu,\xi^\nu\}}$, *where* M_0 *depends on* α, β, μ, *and* ν *only.*

In Theorem 1 we have $\mu < \alpha$ and $\nu < \beta$. A comparable statement can be made for the case $\mu = \alpha$ or $\nu = \beta$; this statement, however, involves certain interpolation spaces that cannot be characterized as spaces of Hölder continuous functions (see Sections 3–5).

Note also that the assumption $h_1 \in \mathcal{C}^{\beta'}([0,\xi];\mathbb{C})$ in (ii) is necessary for (iv) to hold and is only introduced so that $D_x^\beta h_1$ is well defined. The same comment applies to h_2, i.e., to (iii) and (v).

If $h_1 = h_2 = 0$ and $\frac{\mu}{\alpha} \neq \frac{\nu}{\beta}$, then either (iv) or (v) requires more smoothness of f on the boundary than what follows from (i).

In Theorem 1 we restrict ourselves to the case where $f(0,0) = 0$; in fact it turns out that the above results do not hold if this condition is violated. For the purpose of analyzing the case where $f(0,0) \neq 0$ we first study the fundamental solution of (1) and obtain rather precise estimates.

Theorem 2. *Let* α *and* $\beta \in (0,1)$ *and let*

$$(6)\quad \psi_{\alpha,\beta}(t,x) = \int_0^\infty \tau^{-\frac{1}{\alpha}-\frac{1}{\beta}} \varphi_\alpha\Big(t\tau^{-\frac{1}{\alpha}}\Big)\varphi_\beta\Big(x\tau^{-\frac{1}{\beta}}\Big)\,\mathrm{d}\tau, \quad t,x \in \mathbb{R},$$

where

$$(7)\quad \begin{aligned} &\varphi_\mu(t) = \frac{1}{\pi}\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k!}\sin(\pi\mu k)\Gamma(\mu k+1)t^{-\mu k-1}, \quad t>0, \\ &\varphi_\mu(t) = 0, \quad t \le 0, \end{aligned} \qquad \mu \in (0,1).$$

Then $\psi_{\alpha,\beta}$ *is nonnegative, locally integrable on* $\mathbb{R}^2$, *and satisfies*

$$D_t^\alpha \psi_{\alpha,\beta} + D_x^\beta \psi_{\alpha,\beta} = \delta,$$

in the distribution sense, where δ *denotes the Dirac measure in* $\mathbb{R}^2$.

Moreover, there is a constant $C(\alpha,\beta)$ *such that*

$$\begin{aligned} &\frac{1}{C(\alpha,\beta)} t^{-\alpha-1}x^{-\beta-1}\min\{t^{3\alpha},x^{3\beta}\} \\ &\quad \le \psi_{\alpha,\beta}(t,x) \\ &\qquad \le C(\alpha,\beta)t^{-\alpha-1}x^{-\beta-1}\min\{t^{3\alpha},x^{3\beta}\}, \quad t,x>0. \end{aligned}$$

Next we use this fundamental solution to study the solution of the problem

$$D_t^\alpha u + D_x^\beta u = 1, \tag{8}$$

on $(0,\infty)\times(0,\infty)$.

Proposition 3. *Let α and $\beta \in (0,1)$, let $u(\underline{t},\underline{x}) = \int_0^{\underline{t}}\int_0^{\underline{x}} \psi_{\alpha,\beta}(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau$ be the solution of* (8)*, and let $v = D_t^\alpha u$ and $w = D_x^\beta u$. Then v and w are nonnegative, $v(t,0) = w(0,x) = 0$ when t, $x \ge 0$ and $v(0,x) = w(t,0) = 1$ when t, $x > 0$, and there is a constant $C(\alpha,\beta)$ such that*

$$\begin{aligned}
&\frac{1}{C(\alpha,\beta)} t^{-\alpha}x^{-\beta-1}\min\{t^{2\alpha},x^{2\beta}\}\\
&\qquad \le v_x(t,x) = -w_x(t,x)\\
&\qquad\qquad \le C(\alpha,\beta)t^{-\alpha}x^{-\beta-1}\min\{t^{2\alpha},x^{2\beta}\}, \quad t,x>0,\\
&\frac{1}{C(\alpha,\beta)} t^{-\alpha-1}x^{-\beta}\min\{t^{2\alpha},x^{2\beta}\}\\
&\qquad \le -v_t(t,x) = w_t(t,x)\\
&\qquad\qquad \le C(\alpha,\beta)t^{-\alpha-1}x^{-\beta}\min\{t^{2\alpha},x^{2\beta}\}, \quad t,x>0.
\end{aligned} \tag{9}$$

Furthermore,

$$\underline{t}^\mu \underline{x}^\nu v(\underline{t},\underline{x}) \text{ and } \underline{t}^\mu \underline{x}^\nu w(\underline{t},\underline{x}) \in \mathcal{C}^{\mu,\nu}([0,\tau]\times[0,\xi];\mathbb{R}), \tag{10}$$

for all μ, $\nu \in (0,1)$ and τ, $\xi > 0$.

We need the following simple result about Hölder continuous functions of two variables. (The proof is left to the reader.)

Lemma 4. *Let $\mu,\nu \in (0,1]$, τ, $\xi > 0$, let $f : [0,\tau]\times[0,\xi] \to \mathbb{C}$ and define $F : [0,\tau] \to ([0,\xi] \to \mathbb{C})$ by $F(t) = f(t,\underline{x})$ for $t \in [0,\tau]$. Then $f \in \mathcal{C}^{\mu,\nu}([0,\tau]\times[0,\xi];\mathbb{C})$ iff $F \in \mathcal{C}^\mu([0,\tau];\mathcal{B}([0,\xi];\mathbb{C})) \cap \mathcal{B}([0,\tau];\mathcal{C}^\nu([0,\xi];\mathbb{C}))$ and*

$$\begin{aligned}
\|f\|_{\mathcal{C}^{\mu,\nu}([0,\tau]\times[0,\xi])} &\le \big(\|F\|_{\mathcal{C}^\mu([0,\tau];\mathcal{B}([0,\xi];\mathbb{C}))} + \|F\|_{\mathcal{B}([0,\tau];\mathcal{C}^\nu([0,\xi];\mathbb{C}))}\big)\\
&\le 2\|f\|_{\mathcal{C}^{\mu,\nu}([0,\tau]\times[0,\xi])}.
\end{aligned}$$

3. An abstract fractional evolution equation

Let X be a complex Banach space and let $\mathcal{L}(X)$ be the space of bounded linear operators on X. Let B be a closed (not necessarily densely defined) linear map of $\mathcal{D}(B) \subset X$ into X. Thus $\mathcal{D}(B)$ is a Banach space with the graph norm $\|x\|_X + \|Bx\|_X$. The operator B is said to be positive (see [16, p. 91]), if $\rho(-B)$, the resolvent set of $-B$, contains $\mathbb{R}^+ = [0,\infty)$ and

$$\sup_{\lambda\ge 0}\big\|(\lambda+1)(\lambda I+B)^{-1}\big\|_{\mathcal{L}(X)} < \infty.$$

For $\omega \in [0,\pi)$, we define

$$\Sigma_\omega \overset{\text{def}}{=} \{\, \lambda \in \mathbb{C}\setminus\{0\} \mid |\arg\lambda| < \omega \,\}.$$

We recall that, if B is positive, then there exists a number $\eta \in (0,\pi)$ such that $\rho(-B) \supset \overline{\Sigma_\eta}$ and

$$\sup_{\lambda\in\overline{\Sigma_\eta}}\big\|(\lambda+1)(\lambda I+B)^{-1}\big\|_{\mathcal{L}(X)} < \infty. \tag{11}$$

The spectral angle of B is defined by

$$\phi_B \stackrel{\text{def}}{=} \inf\big\{\, \omega \in (0,\pi] \;\big|\; \rho(-B) \supset \overline{\Sigma_{\pi-\omega}} \quad \text{and} \quad \sup_{\lambda\in\overline{\Sigma_{\pi-\omega}}} \big\|(\lambda+1)(\lambda I+B)^{-1}\big\|_{\mathcal{L}(X)} < \infty \,\big\}.$$

Consider the abstract fractional evolution equation

$$\big(D_t^\alpha(u-u_0)\big)(t) + Bu(t) = f(t), \quad t \geq 0, \tag{12}$$

where $\alpha \in (0,1)$, B is a positive operator on X, $u_0 \in X$, and $f \in \mathcal{C}([0,T];X)$, for some $T > 0$.

Definition 5. A function $u : [0,T] \to X$ is said to be a strict solution of (12) on $[0,T]$ if $u \in \mathcal{C}([0,T];\mathcal{D}(B))$, $g_{1-\alpha} * (u-u_0) \in \mathcal{C}^1([0,T];X)$ ($*$ denotes convolution), and (12) holds for all $t \in [0,T]$.

As in Section 1, we observe that if u is a strict solution, then $u(0) = u_0$, hence $u_0 \in \mathcal{D}(B)$.

In this section we state some existence, uniqueness, and regularity results concerning strict solutions of (12). Our main result is Theorem 6 below. Since the formulation of this theorem involves some interpolation spaces, we first recall some properties of these for the sake of completeness. As a general reference on interpolation spaces, see e.g. [12] or [16].

Let B be a positive operator in X; let $\gamma \in (0,1]$ and $p \in [1,\infty]$. We use the notation

$$\mathcal{D}_B(\gamma,p) \stackrel{\text{def}}{=} (X,\mathcal{D}(B))_{\gamma,p},$$
$$\mathcal{D}_B(\gamma) \stackrel{\text{def}}{=} (X,\mathcal{D}(B))_{\gamma}.$$

By [10, Thm. 3.1, p. 159] and [11, p. 314] one has the following characterization of $\mathcal{D}_B(\gamma,\infty)$ and $\mathcal{D}_B(\gamma)$ for $\gamma \in (0,1]$: If η is some number such that $0 \leq \eta < \pi - \phi_B$, then

$$\mathcal{D}_B(\gamma,\infty) = \Big\{\, x \in X \;\Big|\; \sup_{\substack{|\arg\lambda|\leq\eta \\ \lambda\neq 0}} \|\lambda^\gamma B(\lambda I+B)^{-1}x\|_X < \infty \,\Big\},$$
$$\mathcal{D}_B(\gamma) = \Big\{\, x \in \mathcal{D}_B(\gamma,\infty) \;\Big|\; \lim_{\substack{|\lambda|\to\infty \\ |\arg\lambda|\leq\eta}} \|\lambda^\gamma B(\lambda I+B)^{-1}x\|_X = 0 \,\Big\}.$$

Moreover, we can (using the fact that B is invertible), for each η satisfying $0 \leq \eta < \pi - \phi_B$, take

$$\|x\|_{\gamma,\eta} \stackrel{\text{def}}{=} \sup_{\substack{|\arg\lambda|\leq\eta \\ \lambda\neq 0}} \|\lambda^\gamma B(\lambda I+B)^{-1}x\|_X,$$

as the norm in $\mathcal{D}_B(\gamma,\infty)$ and $\mathcal{D}_B(\gamma)$. Finally we recall that when $\gamma \in (0,1]$ one has $\mathcal{D}_B(\gamma) \subset \mathcal{D}_B(\gamma,\infty) \subset \overline{\mathcal{D}(B)}$, $\mathcal{D}_B(1,\infty) \subset \mathcal{D}_B(\gamma,\infty)$, and $\mathcal{D}_B(1) = \{0\}$ (which is easiest to see from the definition of $(X,\mathcal{D}(B))_1$).

Now we can state our main result.

Theorem 6. *Suppose*

(i) $\alpha \in (0,1)$*;*
(ii) B *is a positive operator in a complex Banach space* X *with spectral angle* $\phi_B < \pi(1-\frac{\alpha}{2})$*;*
(iii) $u_0 \in \mathcal{D}(B)$*;*

(iv) $f \in \mathcal{C}([0,T];X)$ *where* $T > 0$.

Then the following statements hold:

(a) *Let* $\gamma \in (0,\alpha]$ *and* $f \in \mathcal{C}^\gamma([0,T];X)$. *Then there is a unique strict solution* u *of* (12) *satisfying* $Bu(\underline{t}) \in \mathcal{C}^\gamma([0,T];X)$ *iff* $Bu_0 - f(0) \in \mathcal{D}_B(\frac{\gamma}{\alpha},\infty)$. *Moreover, in this case there is a constant* $M = M(\gamma,\alpha,B,T)$ *such that*

$$\|Bu(\underline{t}) - f(0)\|_{\mathcal{C}^\gamma([0,T];X)} \leq M\Bigg(\|Bu_0 - f(0)\|_{\mathcal{D}_B(\gamma/\alpha,\infty)} + \|f(\underline{t}) - f(0)\|_{\mathcal{C}^\gamma([0,T];X)}\Bigg).$$

(b) *Let* $\gamma \in (\alpha,1)$ *and* $f \in \mathcal{C}^\gamma([0,T];X)$. *Then there is a unique strict solution* u *of* (12) *satisfying* $Bu(\underline{t}) \in \mathcal{C}^\gamma([0,T];X)$ *iff* $Bu_0 = f(0)$. *Moreover, in this case there is a constant* $M = M(\gamma,\alpha,B,T)$ *such that*

$$\|Bu(\underline{t}) - f(0)\|_{\mathcal{C}^\gamma([0,T];X)} \leq M\|f(\underline{t}) - f(0)\|_{\mathcal{C}^\gamma([0,T];X)}.$$

(c) *Let* $\gamma \in (0,1)$ *and* $f \in \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$. *Then there is a unique strict solution* u *of* (12) *satisfying* $Bu(\underline{t}) \in \mathcal{C}([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$ *iff* $Bu_0 - f(0) \in \mathcal{D}_B(\gamma,\infty)$. *Moreover, in this case there is a constant* $M = M(\gamma,\alpha,B,T)$ *such that*

$$\|Bu(\underline{t}) - f(0)\|_{\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))} \leq M\Bigg(\|Bu_0 - f(0)\|_{\mathcal{D}_B(\gamma,\infty)} + \|f(\underline{t}) - f(0)\|_{\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))}\Bigg).$$

(d) *Let* $\gamma \in (0,\alpha)$ *and* $f \in h^\gamma([0,T];X)$. *Then there is a unique strict solution* u *of* (12) *satisfying* $Bu(\underline{t}) \in h^\gamma([0,T];X)$ *iff* $Bu_0 - f(0) \in \mathcal{D}_B(\frac{\gamma}{\alpha})$.

(e) *Let* $\gamma \in [\alpha,1)$ *and* $f \in h^\gamma([0,T];X)$. *Then there is a unique strict solution* u *of* (12) *satisfying* $Bu(\underline{t}) \in h^\gamma([0,T];X)$ *iff* $Bu_0 = f(0)$.

(f) *Let* $\gamma \in (0,1)$ *and* $f \in \mathcal{C}([0,T];\mathcal{D}_B(\gamma))$. *Then there is a unique strict solution* u *of* (12) *satisfying* $Bu(\underline{t}) \in \mathcal{C}([0,T];\mathcal{D}_B(\gamma))$ *iff* $Bu_0 - f(0) \in \mathcal{D}_B(\gamma)$.

In order to solve (12), we write $u = v + w + B^{-1}f(0)$ where v satisfies

$$D_t^\alpha(v - v_0)(t) + Bv(t) = 0, \quad t \geq 0, \tag{13}$$

with

$$v_0 = u_0 - B^{-1}f(0),$$

and where w is a solution of

$$D_t^\alpha w(t) + Bw(t) = h(t), \quad t \geq 0, \tag{14}$$

with

$$h(\underline{t}) = f(\underline{t}) - f(0).$$

Equations (13) and (14) will be analyzed by different methods.

For the analysis of equation (13) we shall, as mentioned in Section 1, use the resolvent associated with it. As in [6] and [14, p. 52] (where the operator B is densely defined) we set

$$\begin{aligned} S(t)v_0 &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{1,\theta}} \mathrm{e}^{\lambda t}(\lambda^\alpha I + B)^{-1}\lambda^{\alpha-1}v_0\,\mathrm{d}\lambda, \quad t > 0, \\ S(0)v_0 &= v_0, \end{aligned} \tag{15}$$

where

$$\theta \in \left(\frac{\pi}{2}, \min\left\{\pi, \frac{\pi - \phi_B}{\alpha}\right\}\right). \tag{16}$$

(which is possible, provided (ii) of Theorem 6 holds) and

$$\Gamma_{r,\theta} \stackrel{\text{def}}{=} \{\, r\mathrm{e}^{\mathrm{i}t} \mid |t| \le \theta \,\} \cup \{\, \rho \mathrm{e}^{\mathrm{i}\theta} \mid r < \rho < \infty \,\} \cup \{\, \rho \mathrm{e}^{-\mathrm{i}\theta} \mid r < \rho < \infty \,\}. \tag{17}$$

(The orientation of this contour is such that the argument does not decrease along it.) The motivation for this definition is the fact that $(\underline{\lambda}^{\alpha} I + B)^{-1} \underline{\lambda}^{\alpha-1} v_0$ is, formally, the Laplace transform of the solution v of (13).

The following result can, in the case where B is densely defined, essentially be found in e.g. [6] or [14, Sec. 2], and the proof is the same for the case where B is not densely defined.

Lemma 7. *Assume that* (i) *and* (ii) *of Theorem 6, and* (16) *hold. If S is defined by* (15)*, then*

(a) *$S(t) \in \mathcal{L}(X)$ for each $t \ge 0$ and $\sup_{t\ge 0} \|S(t)\|_{\mathcal{L}(X)} < \infty$;*
(b) *$\lim_{t\downarrow 0} \|S(t)v_0 - v_0\|_X = 0$ for each $v_0 \in \overline{\mathcal{D}(B)}$;*
(c) *$S(t)v_0 \in \mathcal{D}(B)$ when $v_0 \in X$ and $t > 0$ and $\sup_{t>0} t^{\alpha} \|BS(t)\|_{\mathcal{L}(X)} < \infty$;*
(d) *$S(\cdot)$ and $BS(\cdot)$ can be extended analytically to the sector $|\arg z| < \theta - \frac{\pi}{2}$ and $\sup_{t>0} (t\|S'(t)\|_{\mathcal{L}(X)} + t^{1+\alpha} \|BS'(t)\|_{\mathcal{L}(X)}) < \infty$;*
(e) *$D_t^{\alpha}(S(\bullet)v_0 - v_0)(t) + BS(t)v_0 = 0$ for $t > 0$ and $v_0 \in X$;*
(f) *If $Bv_0 \in \overline{\mathcal{D}(B)}$, then the function $S(\underline{t})v_0$ is a strict solution of* (13) *on $[0, T]$ for every $T > 0$, and if v is a strict solution of* (13) *on $[0, T]$ for some $T > 0$, then $v(t) = S(t)v_0$ for $t \in [0, T]$.*

Turning to equation (14), we shall use the "method of sums" of Da Prato and Grisvard, [5], to solve this equation. The following result is a reformulation of [5, Thm. 3.11]. This reformulation contains the method of sums in a form suitable for our application.

Theorem 8. *Assume that $\tilde{X}$ is a complex Banach space and that*

(i) *$\tilde{A}$ and $\tilde{B}$ are positive operators in $\tilde{X}$ with spectral angles $\phi_{\tilde{A}}$ and $\phi_{\tilde{B}}$, respectively, such that $\phi_{\tilde{A}} + \phi_{\tilde{B}} < \pi$;*
(ii) *$(\lambda I - \tilde{A})^{-1}(\mu I - \tilde{B})^{-1} = (\mu I - \tilde{B})^{-1}(\lambda I - \tilde{A})^{-1}$ for all $\lambda \in \rho(\tilde{A})$ and $\mu \in \rho(\tilde{B})$.*

If $\tilde{Y}$ is one of the spaces $\mathcal{D}_{\tilde{A}}(\gamma, p)$, $\mathcal{D}_{\tilde{A}}(\gamma)$, $\mathcal{D}_{\tilde{B}}(\gamma, p)$, or $\mathcal{D}_{\tilde{B}}(\gamma)$, where $\gamma \in (0, 1)$ and $p \in [1, \infty]$, and if $y \in \tilde{Y}$, then there is a unique $x \in \mathcal{D}(\tilde{A}) \cap \mathcal{D}(\tilde{B})$ such that $\tilde{A}x + \tilde{B}x = y$. Moreover, $\tilde{A}x$ and $\tilde{B}x \in \tilde{Y}$ and there is a constant C such that

$$\|\tilde{A}x\|_{\tilde{Y}} + \|\tilde{B}x\|_{\tilde{Y}} \le C \|y\|_{\tilde{Y}}.$$

The invertibility of $\tilde{A}$ and $\tilde{B}$ allows us to take λ in [5, Thm. 3.11] to be 0. The fact that not only $\mathcal{D}_{\tilde{A}}(\gamma, p)$ and $\mathcal{D}_{\tilde{B}}(\gamma, p)$ but also $\mathcal{D}_{\tilde{A}}(\gamma)$ and $\mathcal{D}_{\tilde{B}}(\gamma)$ are maximal regularity spaces follows directly from the proof of [5, Thm. 3.11]. Moreover, note that neither A nor B need in fact be densely defined.

In order to apply Theorem 8 to (14) we let $T > 0$ and define the space $\tilde{X}$ to be

$$\tilde{X} \stackrel{\text{def}}{=} \mathcal{C}_{0\mapsto 0}([0, T]; X) \stackrel{\text{def}}{=} \{\, u \in \mathcal{C}([0, T]; X) \mid u(0) = 0 \,\},$$

equipped with the sup-norm. Observe that $h \in \tilde{X}$. Next we define the operator $\tilde{B}$ in $\tilde{X}$ by

$$\begin{aligned} \mathcal{D}(\tilde{B}) &= \mathcal{C}_{0\mapsto 0}([0,T];\mathcal{D}(B)), \\ (\tilde{B}u)(\underline{t}) &= Bu(\underline{t}), \quad u \in \mathcal{D}(\tilde{B}). \end{aligned} \tag{18}$$

We also define the operator $\tilde{A}_\alpha$ in $\tilde{X}$ by

$$\begin{aligned} &\mathcal{D}(\tilde{A}_\alpha) \stackrel{\text{def}}{=} \{\, u \in \mathcal{C}_{0\mapsto 0}([0,T];X) \mid g_{1-\alpha} * u \in \mathcal{C}^1([0,T];X),\ (D_t^\alpha u)(0) = 0 \,\}, \\ &\tilde{A}_\alpha u = D_t^\alpha u, \quad u \in \mathcal{D}(\tilde{A}_\alpha), \end{aligned} \tag{19}$$

when $\alpha \in (0,1)$ and

$$\begin{aligned} \mathcal{D}(\tilde{A}_1) &\stackrel{\text{def}}{=} \{\, u \in \mathcal{C}^1_{0\mapsto 0}([0,T];X) \mid u'(0) = 0 \,\}, \\ \tilde{A}_1 u &= u', \quad u \in \mathcal{D}(\tilde{A}_1). \end{aligned} \tag{20}$$

Equation (14) can now be rewritten as

$$\tilde{A}_\alpha w + \tilde{B} w = h. \tag{21}$$

We must verify that $\tilde{A}_\alpha$ and $\tilde{B}$ satisfy the assumptions of Theorem 8 and we need characterizations of the spaces $\mathcal{D}_{\tilde{B}}(\gamma,\infty)$, $\mathcal{D}_{\tilde{B}}(\gamma)$, $\mathcal{D}_{\tilde{A}_\alpha}(\gamma,\infty)$, and $\mathcal{D}_{\tilde{A}_\alpha}(\gamma)$. First we study the operator $\tilde{B}$.

Lemma 9. *Assume that B is a positive operator in a complex Banach space X with spectral angle ϕ_B and let $\tilde{B}$ be the operator in $\mathcal{C}_{0\mapsto 0}([0,T];X)$ defined by* (18) *for some $T > 0$. Then*

(a) *$\tilde{B}$ is a positive operator with spectral angle $\phi_{\tilde{B}} = \phi_B$;*
(b) *$\mathcal{D}_{\tilde{B}}(\gamma,\infty) = \mathcal{C}_{0\mapsto 0}([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$ when $\gamma \in (0,1]$;*
(c) *$\mathcal{D}_{\tilde{B}}(\gamma) = \mathcal{C}_{0\mapsto 0}([0,T];\mathcal{D}_B(\gamma))$ when $\gamma \in (0,1]$.*

In order to characterize the spaces $\mathcal{D}_{\tilde{A}_\alpha}(\gamma,\infty)$ and $\mathcal{D}_{\tilde{A}_\alpha}(\gamma)$ we need the fact that $\tilde{A}_\alpha$ is a fractional power of $\tilde{A}_1$. This we formulate in Lemma 11. In Theorem 10 (see [16, Sect. 1.15]) we have collected the properties of fractional powers that we need (for simplicity we only consider real powers). In particular, we make use of the reiteration theorem as applied to fractional powers, i.e., case (d) below.

Theorem 10. *Let $\tilde{X}$ be a complex Banach space and let $\tilde{A}$ be a positive, densely defined operator in $\tilde{X}$. Then the following conclusions hold.*

(a) *If $\beta \in \mathbb{R}$ and m and n are integers such that $m > \beta + n > 0$, then the operator*

$$x \in \mathcal{D}(\tilde{A}^m) \mapsto \frac{\Gamma(m)}{\Gamma(\beta+n)\Gamma(m-n-\beta)} \int_0^\infty s^{\beta+n-1} \tilde{A}^{m-n} (sI + \tilde{A})^{-m} x \,\mathrm{d}s \tag{22}$$

is closable, and its closure $\tilde{A}^\beta$ agrees with the usual definition when β is an integer.

(b) *Let σ and $\sigma + \tau \geq 0$. Then $\tilde{A}^\sigma \tilde{A}^\tau = \tilde{A}^{\sigma+\tau}$.*
(c) *Let $\sigma > 0$, $\tau < \sigma$, $\eta \in (\max\{0, \frac{\tau}{\sigma}\}, 1)$, and let $p \in [1,\infty]$. Then $\tilde{A}^\tau$ is an isomorphism from $\mathcal{D}(\tilde{A}^\sigma)$ onto $\mathcal{D}(\tilde{A}^{\sigma-\tau})$, from $\mathcal{D}_{\tilde{A}^\sigma}(\eta,p)$ onto $\mathcal{D}_{\tilde{A}^{\sigma-\frac{\tau}{\eta}}}(\eta,p)$, and from $\mathcal{D}_{\tilde{A}^\sigma}(\eta)$ onto $\mathcal{D}_{\tilde{A}^{\sigma-\frac{\tau}{\eta}}}(\eta)$,*
(d) *Let $0 < \sigma < \tau$ and let $\eta \in (0,1)$ and $p \in [1,\infty]$. Then*

$$\mathcal{D}_{\tilde{A}^\sigma}(\eta,p) = \mathcal{D}_{\tilde{A}^\tau}(\tfrac{\sigma\eta}{\tau},p) \quad \text{and} \quad \mathcal{D}_{\tilde{A}^\sigma}(\eta) = \mathcal{D}_{\tilde{A}^\tau}(\tfrac{\sigma\eta}{\tau}).$$

Case (a) follows from [16, pp. 98–99], case (b) is a special case of [16, formula (2), p. 101], the first part of (c) is a special case of the first part of [16, case (e), p. 101], and the first part of (d) is a special case of [16, case (f), p. 101]. The second part of (c) follows from the second part of [16, case (e), p. 101] combined with the first part of (d), and the last parts of (c) of (d) follow from the already established results with $p = \infty$ by the fact that $\mathcal{D}_{A^\sigma}(\eta)$ is the closure of $\mathcal{D}(A^\mu)$, $\mu \geq \sigma$, in $\mathcal{D}_{A^\sigma}(\eta, \infty)$.

Our next result shows that the operators $\tilde{A}_\alpha$ are indeed fractional powers of $\tilde{A}_1$. This fact allows us to characterize the spaces $\mathcal{D}_{\tilde{A}_\alpha}(\eta, \infty)$.

Lemma 11. *Let X be a complex Banach space, $T > 0$, $\alpha \in (0,1)$, and let $\tilde{A}_\alpha$ and $\tilde{A}_1$ be the operators in $\mathcal{C}_{0\mapsto 0}([0,T];X)$ defined by* (19) *and* (20). *Then*

(a) *$\tilde{A}_1$ is a positive, densely defined operator and $\tilde{A}_1^\alpha = \tilde{A}_\alpha$.*
(b) *$\tilde{A}_\alpha$ is a positive densely defined operator with spectral angle $\phi_{\tilde{A}_\alpha} = \frac{1}{2}\pi\alpha$.*
(c) *$\mathcal{D}_{\tilde{A}_\alpha}(\eta, \infty) = \mathcal{C}^{\alpha\eta}_{0\mapsto 0}([0,T];X)$, $\eta \in (0,1)$.*
(d) *$\mathcal{D}_{\tilde{A}_\alpha}(\eta) = h^{\alpha\eta}_{0\mapsto 0}([0,T];X)$, $\eta \in (0,1)$.*

4. Proofs

The proof of Theorem 6 proceeds as follows. We first prove Lemmas 9 and 11, already stated in Section 3. Next, in Lemma 12, we give some regularity results on (13). The regularity results on (14) are formulated in Lemma 13. The proof of this lemma is basically an application of Theorem 8. Finally, Theorem 6 follows easily from Lemmas 12 and 13.

Proof of 9. (a) Since $((\lambda I + \tilde{B})^{-1}u)(t) = (\lambda I + B)^{-1}(u(t))$ for $t \in [0,T]$, it is clear that $\tilde{B}$ is a positive operator with the same spectral angle as B.

(b) Assume that $u \in \mathcal{D}_{\tilde{B}}(\gamma, \infty)$. Then $u \in \mathcal{C}_{0\mapsto 0}([0,T];X)$ and there is a constant M such that

$$\sup_{\lambda > 0} \|\lambda^\gamma \tilde{B}(\lambda I + \tilde{B})^{-1}u\|_{\mathcal{C}([0,T];X)} \leq M.$$

It follows that

$$\sup_{0 \leq t \leq T} \sup_{\lambda > 0} \|\lambda^\gamma B(\lambda I + B)^{-1}u(t)\|_X \leq M,$$

and so

$$\|u(t)\|_{\mathcal{D}_B(\gamma,\infty)} \leq M, \quad 0 \leq t \leq T.$$

Thus $u \in \mathcal{C}_{0\mapsto 0}([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$.

For the converse, assume that $u \in \mathcal{C}_{0\mapsto 0}([0,T];X)$ and define the number M by $M \stackrel{\text{def}}{=} \sup_{0 \leq t \leq T} \sup_{\lambda > 0} \|\lambda^\gamma B(\lambda I + B)^{-1}u(t)\|_X < \infty$. Then

$$\sup_{\lambda > 0} \|\lambda^\gamma \tilde{B}(\lambda I + \tilde{B})^{-1}u\|_{\mathcal{C}([0,T];X)} = M,$$

i.e., $u \in \mathcal{D}_{\tilde{B}(\gamma,\infty)}$.

(c) First let $u \in \mathcal{C}_{0\mapsto 0}([0,T];\mathcal{D}_B(\gamma))$. By the fact that u takes values in $\mathcal{D}_B(\gamma)$, and is continuous in the norm of $\mathcal{D}_B(\gamma,\infty)$, we have that

$$\lim_{\lambda \to \infty} \|\lambda^\gamma B(\lambda I + B)^{-1}u(t)\|_X = 0,$$

for each $t \in [0,T]$, and that $\sup_{\lambda>0}\|\lambda^\gamma B(\lambda I + B)^{-1}u(\underline{t})\|_X$ is continuous, hence uniformly continuous on $[0,T]$. Combining these two facts we get

$$\lim_{\lambda\to\infty} \sup_{t\in[0,T]} \|\lambda^\gamma B(\lambda I + B)^{-1}u(t)\|_X = 0,$$

which is precisely the statement $\lim_{\lambda\to\infty}\|\lambda^\gamma \tilde{B}(\lambda I + \tilde{B})^{-1}u\|_{\mathcal{C}([0,T];X)} = 0$, and so $u \in \mathcal{D}_{\tilde{B}}(\gamma)$.

Conversely, if $u \in \mathcal{D}_{\tilde{B}}(\gamma)$, then we have $u \in \mathcal{C}_{0\mapsto 0}([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$ by (b). Since $u \in \mathcal{D}_{\tilde{B}}(\gamma)$ it follows easily (see the argument above) that we have $u(t) \in \mathcal{D}_B(\gamma)$ for each $t \in [0,T]$ so that we in fact have $u \in \mathcal{B}([0,T];\mathcal{D}_B(\gamma))$. In order to prove continuity we let s, $t \in [0,T]$ and $\epsilon > 0$ be arbitrary. Then we get for each $\lambda_0 > 0$

$$\begin{aligned}
\|u(t) - u(s)\|_{\mathcal{D}_B(\gamma)} &= \sup_{\lambda>0}\big\|\lambda^\gamma B(\lambda I + B)^{-1}\big(u(t) - u(s)\big)\big\|_X \\
&\le 2 \sup_{\substack{\lambda>\lambda_0\\ \tau\in[0,T]}} \|\lambda^\gamma B(\lambda I + B)^{-1}u(\tau)\|_X \\
&\quad + \sup_{0<\lambda\le\lambda_0} \big\|\lambda^\gamma B(\lambda I + B)^{-1}\big(u(t) - u(s)\big)\big\|_X \\
&\le 2 \sup_{\lambda>\lambda_0} \|\lambda^\gamma \tilde{B}(\lambda I + \tilde{B})^{-1}u\|_{\mathcal{C}([0,T];X)} \\
&\quad + \sup_{0<\lambda\le\lambda_0} \big\|\lambda^\gamma B(\lambda I + B)^{-1}\big\|_{\mathcal{L}(X)}\|u(t) - u(s)\|_X.
\end{aligned} \tag{23}$$

Now we first choose λ_0 so large that the first term on the right-hand side is less than $\epsilon/2$ (which is possible because $u \in \mathcal{D}_{\tilde{B}}(\gamma)$). Then there is, because $u \in \mathcal{C}_{0\mapsto 0}([0,T];X)$, a number $\delta > 0$ such that if $|t - s| < \delta$, then

$$\|u(t) - u(s)\|_X < \frac{\epsilon}{2\sup_{0<\lambda\le\lambda_0}\big\|\lambda^\gamma B(\lambda I + B)^{-1}\big\|_{\mathcal{L}(X)}}.$$

We therefore conclude from (23) and our choice of λ_0 that if $|t - s| < \delta$, then $\|u(t) - u(s)\|_{\mathcal{D}_B(\gamma)} < \epsilon$. □

Proof of Lemma 11. (a) It is clear that $\tilde{A}_1$ is densely defined in $\mathcal{C}_{0\mapsto 0}([0,T];X)$ and we have

$$\big((sI + \tilde{A}_1)^{-1}f\big)(t) = \int_0^t \mathrm{e}^{-s(t-r)}f(r)\,\mathrm{d}r, \quad t \in [0,T].$$

Thus we get, by taking $m = n = 1$ and $\beta = \alpha - 1$ in (22), and using the definition of the gamma function, that

$$\begin{aligned}
(\tilde{A}_1^{\alpha-1}f)(t) &= \tfrac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^\infty s^{\alpha-1} \int_0^t \mathrm{e}^{-s(t-r)}f(r)\,\mathrm{d}r\,\mathrm{d}s \\
&= \int_0^t g_{1-\alpha}(t-r)f(r)\,\mathrm{d}r,
\end{aligned} \tag{24}$$

when $t \in [0,T]$ and $f \in \mathcal{C}_{0\mapsto 0}([0,T];X)$. Now observe that $f \in \mathcal{D}(\tilde{A}_1^\alpha)$ is equivalent to $\tilde{A}_1^{\alpha-1}f \in \mathcal{D}(\tilde{A}_1)$ (see Theorem 10.(c)), which by (24) and by the definition of $\mathcal{D}(\tilde{A}_\alpha)$ is equivalent to $f \in \mathcal{D}(\tilde{A}_\alpha)$. Thus $\mathcal{D}(\tilde{A}_1^\alpha) = \mathcal{D}(\tilde{A}_\alpha)$. Finally by applying $\tilde{A}_1$ to (24) one gets $\tilde{A}_1^\alpha f = \tilde{A}_\alpha f$ for $f \in \mathcal{D}(\tilde{A}_1^\alpha)$.

(b) It is easy to prove that the spectral angle of $\tilde{A}_1$ is at most $\frac{\pi}{2}$. Therefore, by case (a) and [1, Proposition 4.6.10, p. 159] the spectral angle of $\tilde{A}_\alpha$ is at most $\alpha\frac{\pi}{2}$.

In order to show that it is equal to this number, let $x \in X \setminus \{0\}$, $\theta \in (\pi - \alpha\frac{\pi}{2}, \pi)$, $\rho > 0$, $\lambda = \rho e^{i\theta}$, and let $z = \rho^{\frac{1}{\alpha}} e^{i\frac{\theta-\pi}{\alpha}}$. Then a straightforward calculation shows that if $u(\underline{t}) = \frac{1}{z}(e^{z\underline{t}} - 1)x$ and $f(\underline{t}) = -z^{\alpha-1}x - (\int_{\underline{t}}^{\infty} e^{z(t-s)} g_{1-\alpha}(s)\,ds)x$, then u and $f \in \mathcal{C}_{0\mapsto 0}([0,T];X)$ and $\tilde{A}_\alpha u + \lambda u = f$. Moreover, it is easy to see, because $|\arg z| < \frac{\pi}{2}$, that when $\rho \to \infty$ the norm $\|f\|_{\mathcal{C}([0,T];X)}$ remains bounded, but the norm $\|u\|_{\mathcal{C}([0,T];X)}$ grows to infinity. This shows that the spectral angle of $\tilde{A}_\alpha$ is at least $\alpha\frac{\pi}{2}$.

(c) It is well-known, see e.g. [5, Appendix p. 384], that

$$\mathcal{D}_{\tilde{A}_1}(\gamma,\infty) = \mathcal{C}^{\gamma}_{0\mapsto 0}([0,T];X), \quad \gamma \in (0,1). \tag{25}$$

Moreover, by Theorem 10.(d),

$$\mathcal{D}_{\tilde{A}_1^\alpha}(\eta,\infty) = \mathcal{D}_{\tilde{A}_1}(\alpha\eta,\infty),$$

and the claim follows by (25).

(d) Recall that $\mathcal{D}_{\tilde{A}_1}(\gamma)$ is the closure of $\mathcal{D}(\tilde{A}_1)$ in $\mathcal{D}_{\tilde{A}_1}(\gamma,\infty)$ (see [12, Prop. 1.2.12]) and therefore it follows from (25) and [12, Prop. 0.2.1] that

$$\mathcal{D}_{\tilde{A}_1}(\gamma) = h^{\gamma}_{0\mapsto 0}([0,T];X), \quad \gamma \in (0,1). \tag{26}$$

Again we conclude by Theorem 10.(d) that

$$\mathcal{D}_{\tilde{A}_1^\alpha}(\eta) = \mathcal{D}_{\tilde{A}_1}(\alpha\eta),$$

and the claim follows by (26). □

Our next goal is to formulate some regularity results on (13). Observe that by Lemma 7.(d) the function $BS(\underline{t})v_0$ is Lipschitz-continuous on each interval $[T,\infty)$ and therefore we are only interested in the regularity of the solution v of (13) on intervals of the form $[0,T]$ for some $T > 0$.

Lemma 12. *Assume that* (i) *and* (ii) *of Theorem 6, and* (16) *hold and let* $v_0 \in \mathcal{D}(B)$ *and* $v(\underline{t}) = S(\underline{t})v_0$. *Then the following conclusions hold for each* $T > 0$*:*

(a) *Let* $\gamma \in (0,\alpha]$. *Then* $Bv(\underline{t}) \in \mathcal{C}^\gamma([0,T];X)$ *iff* $Bv_0 \in \mathcal{D}_B(\frac{\gamma}{\alpha},\infty)$. *Moreover, in this case there is a constant* $M = M(\gamma,\alpha,B)$ *such that*

$$\|Bv(\underline{t})\|_{\mathcal{C}^\gamma([0,T];X)} \le M\|Bv_0\|_{\mathcal{D}_B(\frac{\gamma}{\alpha},\infty)}. \tag{27}$$

(b) *Let* $\gamma \in (\alpha,1)$. *Then* $Bv(\underline{t}) \in \mathcal{C}^\gamma([0,T];X)$ *iff* $v(\underline{t}) = v_0$ *(iff* $Bv_0 = 0$*).*

(c) *Let* $\gamma \in (0,1]$. *Then* $Bv(\underline{t}) \in \mathcal{C}([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$ *iff* $Bv(\underline{t}) - Bv_0 \in \mathcal{C}([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$ *iff* $Bv_0 \in \mathcal{D}_B(\gamma,\infty)$. *Moreover, in this case there is a constant* $M = M(\gamma,\alpha,B)$ *such that*

$$\|Bv(\underline{t})\|_{\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))} \le M\|Bv_0\|_{\mathcal{D}_B(\gamma,\infty)}. \tag{28}$$

(d) *Let* $\gamma \in (0,\alpha)$. *Then* $Bv(\underline{t}) \in h^\gamma([0,T];X)$ *iff* $Bv_0 \in \mathcal{D}_B(\frac{\gamma}{\alpha})$.

(e) *Let* $\gamma \in [\alpha,1)$. *Then* $Bv(\underline{t}) \in h^\gamma([0,T];X)$ *for some* $T > 0$ *iff* $v(\underline{t}) = v_0$ *(iff* $Bv_0 = 0$*).*

(f) *Let* $\gamma \in (0,1)$. *Then* $Bv(\underline{t}) \in \mathcal{C}([0,T];\mathcal{D}_B(\gamma))$ *iff* $Bv(\underline{t}) - Bv_0 \in \mathcal{C}([0,T];\mathcal{D}_B(\gamma))$ *iff* $Bv_0 \in \mathcal{D}_B(\gamma)$.

Observe that in all cases of Lemma 12, the assumptions made and Lemma 7.(f) imply that $v(\underline{t})$ is a strict solution of (13).

Proof of Lemma 12. (a) Suppose $Bv_0 \in \mathcal{D}_B(\frac{\gamma}{\alpha},\infty)$. Then

$$\sup_{\lambda\in\Gamma_{r,\theta}} \|\lambda^\gamma B(\lambda^\alpha I+B)^{-1}Bv_0\|_X \le \|Bv_0\|_{\mathcal{D}_B(\frac{\gamma}{\alpha},\infty)} < \infty, \quad r\ge 0. \tag{29}$$

Using analyticity to change the integration path and the facts that $\theta \in (\frac{\pi}{2},\pi)$ and $\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{r,\theta}}(\mathrm{e}^{\lambda t}-\mathrm{e}^{\lambda s})\lambda^{-1}\,\mathrm{d}\lambda = 0$ we get from definition (15) for $0<s<t$ and $r>0$

$$\begin{aligned} Bv(t)-Bv(s) &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{1,\theta}} \big(\mathrm{e}^{\lambda t}-\mathrm{e}^{\lambda s}\big)\lambda^{\alpha-1}(\lambda^\alpha I+B)^{-1}Bv_0\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{r,\theta}} \big(\mathrm{e}^{\lambda s}-\mathrm{e}^{\lambda t}\big)\lambda^{-\gamma-1}\lambda^\gamma B(\lambda^\alpha I+B)^{-1}Bv_0\,\mathrm{d}\lambda. \end{aligned}$$

Using (29) and the dominated convergence theorem we can let $r\downarrow 0$ and get, when we change the integration variable,

$$\begin{aligned} Bv(t)-Bv(s) &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{0,\theta}} \big(\mathrm{e}^{\lambda s}-\mathrm{e}^{\lambda t}\big)\lambda^{-\gamma-1}\lambda^\gamma B(\lambda^\alpha I+B)^{-1}Bv_0\,\mathrm{d}\lambda \\ &= (t-s)^\gamma\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_{0,\theta}} \mathrm{e}^{\lambda\frac{s}{t-s}}\big(1-\mathrm{e}^{\lambda}\big)\lambda^{-\gamma-1}\big(\tfrac{\lambda}{t-s}\big)^\gamma B\Big(\big(\tfrac{\lambda}{t-s}\big)^\alpha I+B\Big)^{-1}Bv_0\,\mathrm{d}\lambda. \end{aligned} \tag{30}$$

Since the real part of λ is not greater than 0 on $\Gamma_{0,\theta}$, we see that $|\mathrm{e}^{\lambda\frac{s}{t-s}}|\le 1$ and we get by (29)

$$\begin{aligned} &\|Bv(t)-Bv(s)\|_X \\ &\quad\le |t-s|^\gamma\frac{1}{2\pi}\int_{\Gamma_{0,\theta}} \big|1-\mathrm{e}^{\lambda}\big|\,|\lambda|^{-1-\gamma}\,\mathrm{d}|\lambda|\,\|Bv_0\|_{\mathcal{D}_B(\frac{\gamma}{\alpha},\infty)}, \quad t,s>0. \end{aligned}$$

To see that this inequality holds for $s=0$ it suffices to observe that, by Lemma 7.(f), v is a strict solution; therefore $Bv(\underline{t})\in\mathcal{C}([0,T];X)$. Thus we have obtained the desired conclusion.

Next assume that $Bv(\underline{t})\in\mathcal{C}^\gamma([0,T];X)$ for some $T>0$. It follows from Lemma 7.(c) that $Bv(t)=S(t)Bv_0\in\mathcal{D}(B)$ when $t>0$ and therefore $Bv_0\in\overline{\mathcal{D}(B)}$. This implies by Lemma 7.(a) that one has $Bv(\underline{t})\in\mathcal{B}(\mathbb{R}^+;X)$. Therefore there is a constant c_1 such that when $\lambda>0$

$$\left\|\int_0^\infty \mathrm{e}^{-\lambda t}B(v(t)-v_0)\,\mathrm{d}t\right\|_X \le c_1\int_0^\infty \mathrm{e}^{-\lambda t}t^\gamma\,\mathrm{d}t = c_1\Gamma(\gamma+1)\lambda^{-1-\gamma}.$$

On the other hand,

$$\int_0^\infty \mathrm{e}^{-\lambda t}B(v(t)-v_0)\,\mathrm{d}t = \Big(\lambda^{\alpha-1}(\lambda^\alpha I+B)^{-1}-\lambda^{-1}I\Big)Bv_0,$$

and so

$$\|(\lambda^\alpha)^{\gamma/\alpha}B(\lambda^\alpha I+B)^{-1}Bv_0\|_X \le c_1\Gamma(1+\gamma), \quad \lambda>0. \tag{31}$$

Thus $Bv_0\in\mathcal{D}_B(\frac{\gamma}{\alpha},\infty)$.

(b) Suppose $Bv(\underline{t})\in\mathcal{C}^\gamma([0,T];X)$ for some $\gamma\in(\alpha,1)$. We derive (31) in the same way as above, but this inequality cannot hold for $\gamma>\alpha$ unless $Bv_0=0$ because $\mathcal{D}_B(1)=\{0\}$, and then $v(\underline{t})=v_0$. Conversely, if $v(\underline{t})=v_0$, then $Bv_0=f(0)$ and trivially $Bv(\underline{t})\in\mathcal{C}^\gamma([0,T];X)$.

(c) Suppose that $Bv_0 \in \mathcal{D}_B(\gamma,\infty)$. By Lemma 7.(b), (d) , $Bv(\underline{t}) \in \mathcal{C}([0,T];X)$. From definition (15) (using analyticity to change the integration path) and from the facts that $\theta \in (\frac{\pi}{2},\pi)$ and $\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_{r,\theta}} \mathrm{e}^{\lambda t}\lambda^{-1}\,\mathrm{d}\lambda = 1$ we get for each $t>0$

$$Bv(t) - Bv_0 = -\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_{\frac{1}{t},\theta}} \mathrm{e}^{\lambda t}\lambda^{-1-\gamma}\lambda^{\gamma}B(\lambda^\alpha I + B)^{-1}Bv_0\,\mathrm{d}\lambda. \tag{32}$$

To show that $Bv(\underline{t}) \in \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$ it suffices to observe that for each $t>0$ and $\mu>0$, we get from (32)

$$\begin{aligned}(33)\quad & \mu^\gamma B(\mu I + B)^{-1}B(v(t)-v_0) \\ &= -\frac{1}{2\pi \mathrm{i}}\int_{\Gamma_{1/t,\theta}} \mathrm{e}^{\lambda t}\lambda^{-1}B(\lambda^\alpha I + B)^{-1}\mu^\gamma B(\mu I + B)^{-1}Bv_0\,\mathrm{d}\lambda, \quad t>0,\end{aligned}$$

and to use the facts that $\sup_{|\arg\lambda|\le\theta}\|B(\lambda^\alpha I + B)^{-1}\|_{\mathcal{L}(X)} < \infty$ and that $Bv_0 \in \mathcal{D}_B(\gamma,\infty)$. Thus we also get (28).

If $Bv(\underline{t}) \in \mathcal{C}([0,T];X)\cap\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$, then trivially $Bv_0 \in \mathcal{D}_B(\gamma,\infty)$.

Next, suppose $Bv(\underline{t}) - Bv_0 \in \mathcal{C}([0,T];X)\cap\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$ for some $T>0$. Define

$$F(\mu) \stackrel{\text{def}}{=} \|\mu^\gamma B(\mu I + B)^{-1}Bv_0\|_X,$$
$$G(\mu,t) \stackrel{\text{def}}{=} \|\mu^\gamma B(\mu I + B)^{-1}B(v(t)-v_0)\|_X,$$

where μ and $t>0$. For each μ we have that $F(\mu)$ is well defined and by the assumption, we have $\sup_{\mu>0,0\le t\le T} G(\mu,t) < \infty$. From (33) it follows that there is a constant c_2 such that

$$G(\mu,t) \le c_2 F(\mu), \quad \mu>0, \quad t>0. \tag{34}$$

In the same way as in the second part of case (a) we see that $Bv_0 \in \overline{\mathcal{D}(B)}$ and v is a strict solution of (13). Taking Laplace transforms in (13) (permissible by Lemma 7.(c)) one sees that

$$\int_0^\infty \mathrm{e}^{-\lambda t}(v(t)-v_0)\,\mathrm{d}t = -\lambda^{-1}(\lambda^\alpha I + B)^{-1}Bv_0, \quad \lambda>0,$$

and so

$$Bv_0 = -\int_0^\infty \lambda\mathrm{e}^{-\lambda t}(\lambda^\alpha I + B)(v(t)-v_0)\,\mathrm{d}t, \quad \lambda>0.$$

Apply $\mu^\gamma B(\mu I + B)^{-1}$ to both sides of this equation and split the integral on the right-hand side in two parts to get

$$\begin{aligned}(35)\quad & \mu^\gamma B(\mu I + B)^{-1}Bv_0 \\ &= -\int_0^T \lambda\mathrm{e}^{-\lambda t}(\lambda^\alpha I + B)B^{-1}\mu^\gamma B(\mu I + B)^{-1}B(v(t)-v_0)\,\mathrm{d}t \\ &\quad -\int_T^\infty \lambda\mathrm{e}^{-\lambda t}(\lambda^\alpha I + B)B^{-1}\mu^\gamma B(\mu I + B)^{-1}B(v(t)-v_0)\,\mathrm{d}t.\end{aligned}$$

An estimation of the first integral, I_1, on the right-hand side gives

$$\|I_1\|_X \le c_3(1+\lambda^\alpha),$$

where we have used the uniform boundedness of $G(\mu, t)$ for $\mu > 0$ and $0 \le t \le T$. For the second integral we have by (34)

$$\|I_2\|_X \le c_4 F(\mu) \int_T^\infty \lambda \mathrm{e}^{-\lambda t}(1+\lambda^\alpha)\,\mathrm{d}t \le c_4 F(\mu)\mathrm{e}^{-\lambda T}(1+\lambda^\alpha).$$

Thus we obtain from (35) that

$$F(\mu) \le c_3(1+\lambda^\alpha) + c_4 \mathrm{e}^{-\lambda T}(1+\lambda^\alpha)F(\mu).$$

Choose $\lambda = \lambda_0$ such that $c_4 \mathrm{e}^{-\lambda_0 T}(1+\lambda_0^\alpha) \le \frac{1}{2}$. Then it follows that

$$F(\mu) \le 2c_3(1+\lambda_0^\alpha), \quad \mu > 0,$$

and so $Bv_0 \in \mathcal{D}_B(\gamma, \infty)$.

(d) Suppose $Bv_0 \in \mathcal{D}_B(\frac{\gamma}{\alpha})$ and let $\epsilon > 0$ be arbitrary. Then there are a number $\lambda_\epsilon > 0$ such that

$$\frac{1}{2\pi} \int_{\substack{\Gamma_{0,\theta} \\ |\lambda| \le \lambda_\epsilon}} \left|1 - \mathrm{e}^\lambda\right| |\lambda|^{-1-\gamma}\,\mathrm{d}|\lambda| < \epsilon, \tag{36}$$

and a positive number h_ϵ such that

$$\left\| \left(\tfrac{\lambda}{t-s}\right)^\gamma B \left(\left(\tfrac{\lambda}{t-s}\right)^\alpha I + B \right)^{-1} Bv_0 \right\|_X \le \epsilon, \tag{37}$$

uniformly for $\lambda \in \Gamma_{0,\theta}$ with $|\lambda| \ge \lambda_\epsilon$ and $0 < t - s < h_\epsilon$ with $s, t > 0$. Thus we see from (29), (30), (36), and (37) that

$$\|Bv(t) - Bv(s)\|_X \le \epsilon |t-s|^\gamma \|Bv_0\|_{\mathcal{D}_B(\frac{\gamma}{\alpha})} + \epsilon |t-s|^\gamma \frac{1}{2\pi} \int_{\Gamma_{0,\theta}} \left|1 - \mathrm{e}^\lambda\right| |\lambda|^{-1-\gamma}\,\mathrm{d}|\lambda|,$$

provided $0 < t - s < h_\epsilon$. Finally recall the continuity of $Bv(\underline{s})$ at $s = 0$. This shows that $Bv(\underline{t})$ belongs to the little Hölder space at all points on the positive real axis.

Conversely, suppose that $Bv(\underline{t}) \in h^\gamma([0,T]; X)$ for some $T > 0$. Then one sees, using the fact that $Bv(\underline{t}) \in \mathcal{B}(\mathbb{R}^+; X)$ by Lemma 7.(d), that

$$\begin{aligned} \left\| \int_0^\infty \mathrm{e}^{-\lambda t} B(v(t) - v_0)\,\mathrm{d}t \right\|_X &\le \int_0^\infty c(t) \mathrm{e}^{-\lambda t} t^\gamma\,\mathrm{d}t \\ &= \lambda^{-1-\gamma} \int_0^\infty c\left(\tfrac{s}{\lambda}\right) \mathrm{e}^{-s} s^\gamma\,\mathrm{d}s, \quad \lambda > 0, \end{aligned}$$

where $c(\underline{t})$ is a bounded and continuous function on $\mathbb{R}^+$ with $c(0) = 0$. By this last fact,

$$\lim_{\lambda\to\infty} \int_0^\infty c\left(\tfrac{s}{\lambda}\right) \mathrm{e}^{-s} s^\gamma\,\mathrm{d}s = 0.$$

Thus we can conclude, in the same way as in the proof of case (a), that

$$\lim_{\lambda\to\infty} \|(\lambda^\alpha)^{\gamma/\alpha} B(\lambda^\alpha I + B)^{-1} Bv_0\|_X = 0, \tag{38}$$

and it follows that $Bv_0 \in \mathcal{D}_B(\frac{\gamma}{\alpha})$.

(e) If $Bv(\underline{t}) \in h^\alpha([0,T]; X)$ for some $T > 0$, then we again have (38), but now with $\gamma = \alpha$. However, because $\mathcal{D}_B(1) = \{0\}$, we get $Bv_0 = 0$ and hence $v(\underline{t}) = v_0$. The converse follows immediately.

(f) Let $Bv_0 \in \mathcal{D}_B(\gamma)$. By (c),

$$Bv(\underline{t}) \in \mathcal{C}([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty)).$$

From (33) one immediately gets $Bv(t) \in \mathcal{D}_B(\gamma)$ for each $t > 0$. In order to establish continuity we let s, $t \in [0,T]$ and $\lambda_0 > 0$. Then we have, because $Bv(\underline{t}) = S(\underline{t})Bv_0$,

$$\begin{aligned}\|Bv(t) - Bv(s)\|_{\mathcal{D}_B(\gamma)} &= \sup_{\lambda>0}\big\|\lambda^\gamma B(\lambda I + B)^{-1}\big(Bv(t) - Bv(s)\big)\big\|_X \\ &\le 2 \sup_{\substack{\lambda>\lambda_0\\ \tau\in[0,T]}} \|\lambda^\gamma B(\lambda I + B)^{-1} Bv(\tau)\|_X \\ &\quad + \sup_{0<\lambda\le\lambda_0} \|\lambda^\gamma B(\lambda I + B)^{-1}\|_{\mathcal{L}(X)} \|Bv(t) - Bv(s)\|_X \\ &\le 2 \sup_{\lambda>\lambda_0} \|\lambda^\gamma B(\lambda I + B)^{-1} Bv_0\|_X \sup_{\tau\in[0,T]} \|S(\tau)\|_{\mathcal{L}(X)} \\ &\quad + \sup_{0<\lambda\le\lambda_0} \|\lambda^\gamma B(\lambda I + B)^{-1}\|_{\mathcal{L}(X)} \|Bv(t) - Bv(s)\|_X.\end{aligned}$$

Now we can complete the proof by first choosing λ_0 to be sufficiently large and then take $|t-s|$ to be sufficiently small, in the same way as in the proof of Lemma 9.(c).

If $Bv(\underline{t}) \in \mathcal{C}([0,T];\mathcal{D}_B(\gamma))$, then trivially $Bv_0 \in \mathcal{D}_B(\gamma)$.

Next, assume that $Bv(\underline{t}) - Bv_0 \in \mathcal{C}([0,T];\mathcal{D}_B(\gamma))$ for some $T > 0$. Recall that by the converse part of (c) we then have $Bv_0 \in \mathcal{D}_B(\gamma,\infty)$, and so by the first part of the proof of case (c), $Bv(\underline{t}) \in \mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$. Also note that if $g \in \mathcal{C}([0,T];\mathcal{D}_B(\gamma))$, then for every $\epsilon > 0$ there is a number $\mu_\epsilon < \infty$ such that, for all $t \in [0,T]$,

$$\|\mu^\gamma B(\mu I + B)^{-1} g(t)\|_X \le \epsilon, \quad \mu \ge \mu_\epsilon.$$

Take μ_ϵ as above with $g = B(v - v_0)$ and recall (35) and the fact that $Bv(\underline{t}) \in \mathcal{B}(\mathbb{R}^+;X)$ by Lemma 7 (d). If $\mu \ge \mu_\epsilon$, then

$$\begin{aligned}\|\mu^\gamma B(\mu I + B)^{-1} Bv_0\|_X &\le \epsilon c_5 \int_0^T \lambda(1+\lambda^\alpha)\mathrm{e}^{-\lambda t}\,\mathrm{d}t + c_5 \int_T^\infty \lambda \mathrm{e}^{-\lambda t}(1+\lambda^\alpha)\,\mathrm{d}t \\ &\le \epsilon c_5(1+\lambda^\alpha) + c_5 \mathrm{e}^{-\lambda T}(1+\lambda^\alpha),\end{aligned}$$

where c_5 is independent of λ and μ.

Let $\hat\epsilon > 0$ be arbitrary. Take $\lambda = \lambda_0$ such that $c_5 \mathrm{e}^{-\lambda_0 T}(1+\lambda_0^\alpha) \le \frac{\hat\epsilon}{2}$, and let ϵ be such that $\epsilon c_5(1+\lambda_0^\alpha) \le \frac{\hat\epsilon}{2}$. It follows that

$$\|\mu^\gamma B(\mu I + B)^{-1} Bv_0\|_X \le \hat\epsilon, \quad \mu \ge \mu_\epsilon.$$

Thus $Bv_0 \in \mathcal{D}_B(\gamma)$.

This completes the proof of Lemma 12. ∎

We proceed to the non-homogeneous equation (14).

Lemma 13. *Assume that* (i) *and* (ii) *of Theorem 6, and* (16) *hold and that* $\gamma \in (0,1)$. *Then the following is true:*

(a) *Let* $h \in \mathcal{C}^\gamma_{0\mapsto 0}([0,T];X)$. *Then there is a unique strict solution* w *of* (14) *on* $[0,T]$ *satisfying* $Bw(\underline{t}) \in \mathcal{C}^\gamma_{0\mapsto 0}([0,T];X)$. *Moreover, in this case there is a constant* $M = M(\gamma,\alpha,B,T)$ *such that*

$$\|Bw(\underline{t})\|_{\mathcal{C}^\gamma([0,T];X)} \le M\|h\|_{\mathcal{C}^\gamma([0,T];X)}. \tag{39}$$

(b) *Let* $h \in \mathcal{C}_{0\mapsto 0}([0,T];X)\cap\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$. *Then there is a unique strict solution* w *of* (14) *on* $[0,T]$ *with* $Bw(\underline{t}) \in \mathcal{C}_{0\mapsto 0}([0,T];X)\cap\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))$. *Moreover, in this case there is a constant* $M = M(\gamma,\alpha,B,T)$ *such that*

$$\|Bw(\underline{t})\|_{\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))} \le M\|h\|_{\mathcal{B}([0,T];\mathcal{D}_B(\gamma,\infty))}. \tag{40}$$

(c) *Let* $h \in h^{\gamma}_{0\mapsto 0}([0,T];X)$. *Then there is a unique strict solution* w *of* (14) *on* $[0,T]$ *satisfying* $Bw(\underline{t}) \in h^{\gamma}_{0\mapsto 0}([0,T];X)$.

(d) *Let* $h \in \mathcal{C}_{0\mapsto 0}([0,T];\mathcal{D}_B(\gamma))$. *Then there is a unique strict solution* w *of* (14) *on* $[0,T]$ *satisfying* $Bw(\underline{t}) \in \mathcal{C}_{0\mapsto 0}([0,T];\mathcal{D}_B(\gamma))$.

Proof of Lemma 13. For the proof it suffices to make the following observations.

To obtain (a) and (c) with $\gamma \in (0,\alpha)$ and (b) and (d), it suffices to apply Theorem 8 with the operators $\tilde{A}_\alpha$ and $\tilde{B}$ in $\mathcal{C}_{0\mapsto 0}([0,T];X)$ and to use Lemma 9 (b) and (c) together with Lemma 11 (c) and (d) (in this last lemma, take $\eta = \frac{\gamma}{\alpha}$).

There remain the cases $\gamma \in [\alpha,1)$ in (a) and (c).

Consider (14) with $h \in \mathcal{C}^{\gamma}_{0\mapsto 0}([0,T];X)$ where $\gamma \in [\alpha,1)$. Let $\epsilon > 0$ be such that $0 < \gamma - \epsilon < \alpha$. Apply $\tilde{A}_\epsilon$ to (14); write $g = \tilde{A}_\epsilon h$ and $v = \tilde{A}_\epsilon u$ in order to get

$$\tilde{A}_\alpha v + \tilde{B} v = g.$$

By Theorem 10.(c) and by (c) one has $g \in \mathcal{C}^{\gamma-\epsilon}_{0\mapsto 0}([0,T];X)$. Thus, by the part already proved, $\tilde{B}v \in \mathcal{C}^{\gamma-\epsilon}_{0\mapsto 0}([0,T];X)$. Define $u = A^{-\epsilon}v$. Then $\tilde{B}u = A^{-\epsilon}\tilde{B}v \in \mathcal{C}^{\gamma}_{0\mapsto 0}([0,T];X)$ by Theorem 10.(c) and u solves (14). The same arguments apply in the little Hölder case (c).

An examination of the proof of [5, Thm. 3.11] together with Theorem 10 (needed to handle the case $\gamma \ge \alpha$) gives (39) and (40). □

Proof of Theorem 6. We study case (a) only, because the other cases are quite similar.

Assume first that $Bu_0 - f(0) \in \mathcal{D}_B(\frac{\gamma}{\alpha},\infty)$. Then it follows from Lemmas 12.(a) and 13.(a) that there are solutions v and w of equations (13) and (14), respectively, such that Bv and $Bw \in \mathcal{C}^{\gamma}([0,T];X)$ (in Lemma 12.(a) take $v_0 = u_0 - B^{-1}f(0)$ and in Lemma 13.(a) take $h = f - f(0)$). Therefore $u = v + w + B^{-1}f(0)$ is the desired solution of (12).

Conversely, suppose we have a solution u of (12) such that $Bu \in \mathcal{C}^{\gamma}([0,T];X)$. By Lemma 13.(a) we have a solution w of (14) (where $h = f - f(0)$) such that $Bw \in \mathcal{C}^{\gamma}_{0\mapsto 0}([0,T];X)$. Necessarily, we have $w(0) = 0$ as well. This means that if we let $v = u - w - B^{-1}f(0)$ then v is a solution of (13) (with $v_0 = u_0 - B^{-1}f(0)$) such that $Bv \in \mathcal{C}^{\gamma}([0,T];X)$. Then we can use Lemma 7 (f) and Lemma 12.(a) to conclude that $Bu_0 - f(0) \in \mathcal{D}_B(\frac{\gamma}{\alpha},\infty)$.

The desired inequality is an immediate consequence of (27) and (39) and the uniqueness follows from the uniqueness of the solution of (14). □

5. Proofs for the results on equation (1)

Proof of Theorem 1. We divide the problem in two parts. We try to find solutions $v(\underline{t},\underline{x})$ and $w(\underline{t},\underline{x})$, respectively, of the equations

$$D_t^\alpha\big(v(\bullet,\underline{x})(\underline{t}) - h_1(\underline{x})\big) + D_x^\beta v(\underline{t},\bullet)(\underline{x}) = \tfrac{1}{2}f(\underline{t},\underline{x}) - \tfrac{1}{2}f(\underline{t},0) + \tfrac{1}{2}f(0,\underline{x}), \tag{41}$$

and

$$D_t^\alpha\big(w(\bullet,\underline{x})\big)(\underline{t}) + D_x^\beta\big(w(\underline{t},\bullet) - h_2(\underline{t})\big)(\underline{x}) = \tfrac{1}{2}f(\underline{t},\underline{x}) + \tfrac{1}{2}f(\underline{t},0) - \tfrac{1}{2}f(0,\underline{x}). \tag{42}$$

For (41) we let

$$V(t) = v(t, \underline{x}), \quad t \geq 0,$$
$$F(t) = \tfrac{1}{2}f(t, \underline{x}) - \tfrac{1}{2}f(t, 0) + \tfrac{1}{2}f(0, \underline{x}), \quad t \geq 0, \tag{43}$$
$$V_0 = h_1(\underline{x}),$$

and consider the equation

$$D_t^\alpha (V - V_0)(t) + BV(t) = F(t), \quad t \geq 0, \tag{44}$$

in the space $X = \mathcal{C}_{0\mapsto 0}([0, \xi]; \mathbb{C})$ with B the fractional derivative with respect to x, i.e., $B = D_x^\beta$. In fact, by replacing t by x and T by ξ in (19) we can take $B = \tilde{A}_\beta$.

First we apply Theorem 6.(a) with $\gamma = \mu$. For this we observe that our assumption (i) implies that $F \in \mathcal{C}^\mu([0, \tau]; X)$ and that by Lemma 11.(c) $\mathcal{D}_B(\frac{\mu}{\alpha}, \infty) = \mathcal{C}_{0\mapsto 0}^{\frac{\beta\mu}{\alpha}}([0, \xi]; \mathbb{C})$. Thus by (iv) and by the assumptions $f(0, 0) = (D_x^\beta h_1)(0) = 0$, we have $BV_0 - F(0) \in \mathcal{D}_B(\frac{\mu}{\alpha}, \infty)$. We conclude that there is a strict solution V on $[0, \tau]$ such that

$$BV(\underline{t}) \in \mathcal{C}^\mu([0, \tau]; \mathcal{C}([0, \xi]; \mathbb{C})). \tag{45}$$

If we now define v by (43), we see that v is in fact a solution of (41).

Next we take $\gamma = \frac{\nu}{\beta}$ and apply Theorem 6(c). For this we observe that now $\mathcal{D}_B(\gamma, \infty) = \mathcal{C}_{0\mapsto 0}^\nu([0, \xi]; \mathbb{C})$. Thus, by (i) and Lemma 4, we see that

$$F \in \mathcal{B}([0, \tau]; \mathcal{D}_B(\gamma, \infty))$$

and that, by (iv), $BV_0 - F(0) \in \mathcal{D}_B(\gamma, \infty)$ and we conclude that

$$BV(\underline{t}) \in \mathcal{B}([0, \tau]; \mathcal{C}^\nu([0, \xi]; \mathbb{C})). \tag{46}$$

Now combine (45) and (46) with Lemma 4 to get that

$$D_x^\beta v(\underline{t}, \underline{x}) \in \mathcal{C}^{\mu,\nu}([0, \tau] \times [0, \xi]; \mathbb{C}).$$

The Hölder continuity of the term $D_t^\alpha (v - h_1^\square)$ follows from the assumption that f is Hölder continuous and from the fact that v is a solution of (41).

The results one needs about the function w are obtained in exactly the same manner by interchanging the roles of t and x.

To get the estimates on $D_t^\alpha (u - h_1^\square)$ and on $D_x^\beta (u - h_2^\square)$ we first take $\tau = \xi = 1$ and in this case (5) follows from the estimates one gets from the applications of Theorem 6 above together with Lemma 4 and Lemma 11.(c). Next we take τ and $\xi > 0$ such that $\tau^\alpha = \xi^\beta$ and let u be the solution of (1) on $[0, \tau] \times [0, \xi]$. Let $\delta_{\tau,\xi}$ be the operator $\delta_{\tau,\xi} v(t, x) = v(\tau t, \xi x)$. A simple calculation, using the facts that $D_t^\alpha (\delta_{\tau,\xi} u) = \tau^\alpha \delta_{\tau,\xi}(D_t^\alpha u)$ and $D_x^\beta (\delta_{\tau,\xi} u) = \xi^\beta \delta_{\tau,\xi}(D_t^\beta u)$ and the assumption that $\tau^\alpha = \xi^\beta$, shows that if t, $x \in [0, 1]$, then

$$D_t^\alpha \big(\delta_{\tau,\xi} u - \delta_{\tau,\xi} h_1^\square\big)(t, x) + D_x^\beta \big(\delta_{\tau,\xi} u - \delta_{\tau,\xi} h_2^\square\big)(t, x) = \tau^\alpha (\delta_{\tau,\xi} f)(t, x),$$

so that $\delta_{\tau,\xi} u$ is a solution of an equation of the form (1) on the rectangle $[0, 1] \times [0, 1]$ and we can apply (5) to this case. Furthermore one sees that

$$\min\{1, \tau^\mu, \xi^\nu\} \|v\|_{\mathcal{C}^{\mu,\nu}([0,\tau]\times[0,\xi])} \leq \|\delta_{\tau,\xi} v\|_{\mathcal{C}^{\mu,\nu}([0,1]\times[0,1])} \leq \max\{1, \tau^\mu, \xi^\nu\} \|v\|_{\mathcal{C}^{\mu,\nu}([0,\tau]\times[0,\xi])},$$

with corresponding inequalities for Hölder continuous functions of one variable. Thus we can conclude that (5) holds when $\tau^\alpha = \xi^\beta$. If this is not the case, if for example $\tau^\alpha > \xi^\beta$, then we extend f to the rectangle $[0, \tau] \times [\xi, \tau^{\alpha/\beta}]$ by $f(t, x) =$

$f(t,\xi)$ for $x \geq \xi$ and we extend h_1 so that $f(0,\underline{x}) - D_x^\beta h_1(\underline{x})$ is a constant on $[\xi, \tau^{\alpha,\beta}]$ and use the fact that the norm in $\mathcal{C}^{\mu,\nu}(Q;\mathbb{C})$ is a nondecreasing function of Q (with respect to set inclusion). This completes the proof because the uniqueness of the solution is a consequence of the existence of a fundamental solution (see Theorem 2). □

Proof of Theorem 2. In [13] it is shown that the Laplace transform of φ_μ is given by

$$\widehat{\varphi_\mu}(\underline{z}) = \mathrm{e}^{-\underline{z}^\mu}. \tag{47}$$

The facts that $\underline{z}^\mu$ is a Bernstein function and $\mathrm{e}^{-\underline{x}}$ is completely monotone imply that $\mathrm{e}^{-\underline{z}^\mu}$ is completely monotone (see [14, p. 91]). Therefore, by Bernstein's theorem ([14, p. 90]),

$$\varphi_\mu(\underline{t}) \geq 0, \quad \int_0^\infty \varphi_\mu(t)\,\mathrm{d}t = 1.$$

From (47) one concludes immediately that

$$t\varphi_\mu(t) = \int_0^t \mu g_{1-\mu}(t-s)\varphi_\mu(s)\,\mathrm{d}s, \quad t > 0, \tag{48}$$

and also that if one defines

$$\Phi_\mu(t,\tau) \stackrel{\mathrm{def}}{=} \tau^{-\frac{1}{\mu}}\varphi_\mu(t\tau^{-\frac{1}{\mu}}), \quad t,\tau > 0, \tag{49}$$

then the Laplace transform of Φ_μ with respect to the first variable is

$$\widehat{\Phi_\mu}(\underline{z},\underline{\tau}) = \mathrm{e}^{-\underline{z}^\mu \underline{\tau}}.$$

Now we see from (6) and (49) that $\psi_{\alpha,\beta}$ satisfies

$$\psi_{\alpha,\beta}(t,x) = \int_0^\infty \Phi_\alpha(t,\tau)\Phi_\beta(x,\tau)\,\mathrm{d}\tau, \quad t,x \geq 0.$$

By taking the Laplace transforms with respect to both variables one concludes that

$$\widehat{\psi_{\alpha,\beta}}(\underline{z_t},\underline{z_x}) = \frac{1}{\underline{z_t}^\alpha + \underline{z_x}^\beta},$$

and this implies that $\psi_{\alpha,\beta}$ satisfies

$$D_t^\alpha \psi_{\alpha,\beta} + D_x^\beta \psi_{\alpha,\beta} = \delta,$$

where δ denotes the Dirac measure in $\mathbb{R}^2$.

By (7) and the fact that φ_μ is continuous and strictly positive on $(0,\infty)$ (see (48)) we conclude that there is a constant c_6 depending on μ such that

$$\frac{1}{c_6}t^{-\mu-1} \leq \varphi_\mu(t) \leq c_6 t^{-\mu-1}, \quad t \geq 1. \tag{50}$$

For t close to 0 it follows from [2, formula (25,23), p. 237] that when $\mu \in (0,1)$ there are positive constants c_7 and c_8 such that

$$\mathrm{e}^{-c_7 t^{-\frac{\mu}{1-\mu}}} \leq \varphi_\mu(t) \leq c_7 \mathrm{e}^{-c_8 t^{-\frac{\mu}{1-\mu}}}, \quad t \in (0,1]. \tag{51}$$

Now we use (51) to complete the proof. Suppose that $t^\alpha \geq x^\beta > 0$. Using (50) we easily see that there is a constant c_9 depending on α and β such that

$$\frac{1}{c_9}t^{-\alpha-1}x^{2\beta-1} \leq \int_0^{x^\beta} \tau^{-\frac{1}{\alpha}-\frac{1}{\beta}}\varphi_\alpha(t\tau^{-\frac{1}{\alpha}})\varphi_\beta(x\tau^{-\frac{1}{\beta}})\,\mathrm{d}\tau \leq c_9 t^{-\alpha-1}x^{2\beta-1}. \tag{52}$$

Since the function φ_μ is nonnegative, we see that we have established the claimed lower bound and it suffices to get upper bounds for the integral over $[x^\beta, \infty)$.

In (51) we can choose the constants c_7 and c_8 to be the same for both α and β and then we get

$$\begin{aligned}\int_{x^\beta}^{t^\alpha} \tau^{-\frac{1}{\alpha}-\frac{1}{\beta}}\varphi_\alpha(t\tau^{-\frac{1}{\alpha}})\varphi_\beta(x\tau^{-\frac{1}{\beta}})\,\mathrm{d}\tau &\le c_6c_7t^{-\alpha-1}\int_{x^\beta}^{t^\alpha} \tau^{1-\frac{1}{\beta}}\mathrm{e}^{-c_8x^{-\frac{\beta}{1-\beta}}\tau^{\frac{1}{1-\beta}}}\,\mathrm{d}\tau \\ &= c_6c_7(1-\beta)t^{-\alpha-1}x^{2\beta-1}\int_1^{(t^\alpha/x^\beta)^{\frac{1}{1-\beta}}} s^{2-\frac{1}{\beta}-2\beta}\mathrm{e}^{-c_8s}\,\mathrm{d}s \\ &\le t^{-\alpha-1}x^{2\beta-1}c_6c_7(1-\beta)\int_1^\infty s^{2-\frac{1}{\beta}-2\beta}\mathrm{e}^{-c_8s}\,\mathrm{d}s,\end{aligned}$$

and (substitute $\tau^{\frac{1}{1-\alpha}}t^{-\frac{\alpha}{1-\alpha}} = s$)

$$\begin{aligned}&\int_{t^\alpha}^\infty \tau^{-\frac{1}{\alpha}-\frac{1}{\beta}}\varphi_\alpha(t\tau^{-\frac{1}{\alpha}})\varphi_\beta(x\tau^{-\frac{1}{\beta}})\,\mathrm{d}\tau \\ &\qquad\le c_7^2\int_{\tau^\alpha}^\infty \tau^{-\frac{1}{\alpha}-\frac{1}{\beta}}\mathrm{e}^{-c_8t^{-\frac{\alpha}{1-\alpha}}\tau^{\frac{1}{1-\alpha}}}\,\mathrm{e}^{-c_8x^{-\frac{\beta}{1-\beta}}\tau^{\frac{1}{1-\beta}}}\,\mathrm{d}\tau \\ &\qquad= c_7^2(1-\alpha)t^{\alpha-1-\frac{\alpha}{\beta}}\int_1^\infty s^{-(1-\alpha)(\frac{1}{\alpha}+\frac{1}{\beta})-\alpha}\mathrm{e}^{-c_8s-c_8(t^\alpha/x^\beta)^{\frac{1}{1-\beta}}s^{\frac{1-\alpha}{1-\beta}}}\,\mathrm{d}s.\end{aligned}$$

Now observe that there is a constant c_{10} such that

$$\mathrm{e}^{-c_8\tau} \le c_{10}\tau^{-\frac{(1-\beta)(2\beta-1)}{\beta}}, \quad \tau \ge 1,$$

and that $(t^\alpha/x^\beta)^{\frac{1}{1-\beta}}s^{\frac{1-\alpha}{1-\beta}} \ge 1$ when $s \ge 1$. Then we get

$$\begin{aligned}&\int_{t^\alpha}^\infty \tau^{-\frac{1}{\alpha}-\frac{1}{\beta}}\varphi_\alpha(t\tau^{-\frac{1}{\alpha}})\varphi_\beta(x\tau^{-\frac{1}{\beta}})\,\mathrm{d}\tau \\ &\qquad\qquad\le t^{-\alpha-1}x^{2\beta-1}c_7^2c_{10}(1-\alpha)\int_1^\infty \mathrm{e}^{-c_8s}s^{-\frac{(1-\alpha)(2\beta-1)}{\beta}}\,\mathrm{d}s.\end{aligned}$$

Thus we have established the desired inequality in the case where $t^\alpha \ge x^\beta$. The opposite case is handled analogously. □

Proof of Proposition 3. It is clear that

$$v_x(t,x) = \int_0^t g_{1-\alpha}(t-s)\psi_{\alpha,\beta}(s,x)\,\mathrm{d}s, \quad t,x>0.$$

Using (6), (48), and a change of variable, one thus sees that

$$v_x(t,x) = \frac{t}{\alpha}\int_0^\infty \tau^{-1-\frac{1}{\alpha}-\frac{1}{\beta}}\varphi_\alpha\Big(t\tau^{-\frac{1}{\alpha}}\Big)\varphi_\beta\Big(x\tau^{-\frac{1}{\beta}}\Big)\,\mathrm{d}\tau, \quad t,x>0.$$

Now one can use exactly the same argument as in the proof of Theorem 2 to derive the first inequality in (9). The second inequality is obtained by applying a similar argument to w_t.

It remains to establish (10). Let μ and $\nu \in (0,1)$ be arbitrary and note that it is clearly sufficient to consider the function v only. Let $0 < s < t$ and $x > 0$ be arbitrary. Because $0 < v(s,x) < 1$, we have

$$\frac{|t^\mu x^\nu v(t,x) - s^\mu x^\nu v(s,x)|}{|t-s|^\mu} \le x^\nu + x^\nu\frac{s^\mu|v(s,x)-v(t,x)|}{(t-s)^\mu}.$$

Suppose first that $t^\alpha \le x^\beta$. Then we have by (9)

$$\frac{s^\mu|v(s,x)-v(t,x)|}{(t-s)^\mu} \le \frac{C(\alpha,\beta)}{\alpha}\frac{s^\mu x^{-\beta}(t^\alpha-s^\alpha)}{(t-s)^\mu} = \frac{C(\alpha,\beta)}{\alpha}\frac{\frac{t^\alpha}{x^\beta}((\frac{t}{s})^\alpha-1)}{(\frac{t}{s})^\alpha((\frac{t}{s})-1)^\mu} \le \frac{C(\alpha,\beta)}{\alpha}\sup_{r\ge 1}\frac{r^\alpha-1}{r^\alpha(r-1)^\mu}.$$

Similarly, if $s^\alpha \ge x^\beta$, then we have

$$\frac{s^\mu|v(s,x)-v(t,x)|}{(t-s)^\mu} \le \frac{C(\alpha,\beta)}{\alpha}\frac{s^\mu x^{\beta}(s^{-\alpha}-t^{-\alpha})}{(t-s)^\mu} = \frac{C(\alpha,\beta)}{\alpha}\frac{\frac{x^\beta}{s^\alpha}((\frac{t}{s})^\alpha-1)}{(\frac{t}{s})^\alpha((\frac{t}{s})-1)^\mu} \le \frac{C(\alpha,\beta)}{\alpha}\sup_{r\ge 1}\frac{r^\alpha-1}{r^\alpha(r-1)^\mu}.$$

Finally, if $s^\alpha < x^\beta < t^\alpha$, then we write

$$\frac{s^\mu|v(s,x)-v(t,x)|}{(t-s)^\mu} \le \frac{s^\mu|v(s,x)-v(x^{\frac{\beta}{\alpha}},x)|}{(x^{\frac{\beta}{\alpha}}-s)^\mu} + \frac{(x^{\frac{\beta}{\alpha}})^\mu|v(x^{\frac{\beta}{\alpha}},x)-v(t,x)|}{(t-x^{\frac{\beta}{\alpha}})^\mu},$$

and use the results above. Thus we have proved that

$$\sup_{\substack{t,s\in(0,\tau)\\ t\ne s}}\sup_{x\in[0,\xi]}\frac{|t^\mu x^\nu v(t,x)-s^\mu x^\nu v(s,x)|}{|t-s|^\mu} < \infty.$$

Using the same kind of argument we get

$$\sup_{\substack{x,y\in[0,\xi]\\ x\ne y}}\sup_{t\in[0,\tau]}\frac{|t^\mu x^\nu v(t,x)-t^\mu y^\nu v(t,y)|}{|x-y|^\nu} < \infty.$$

By Lemma 4 we have $\underline{t}^\mu\underline{x}^\nu v(\underline{t},\underline{x}) \in \mathcal{C}^{\mu,\nu}([0,\tau]\times[0,\xi];\mathbb{R})$ and since $v+w=1$, the same conclusion holds with v replaced by w. This completes the proof. $\square$

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